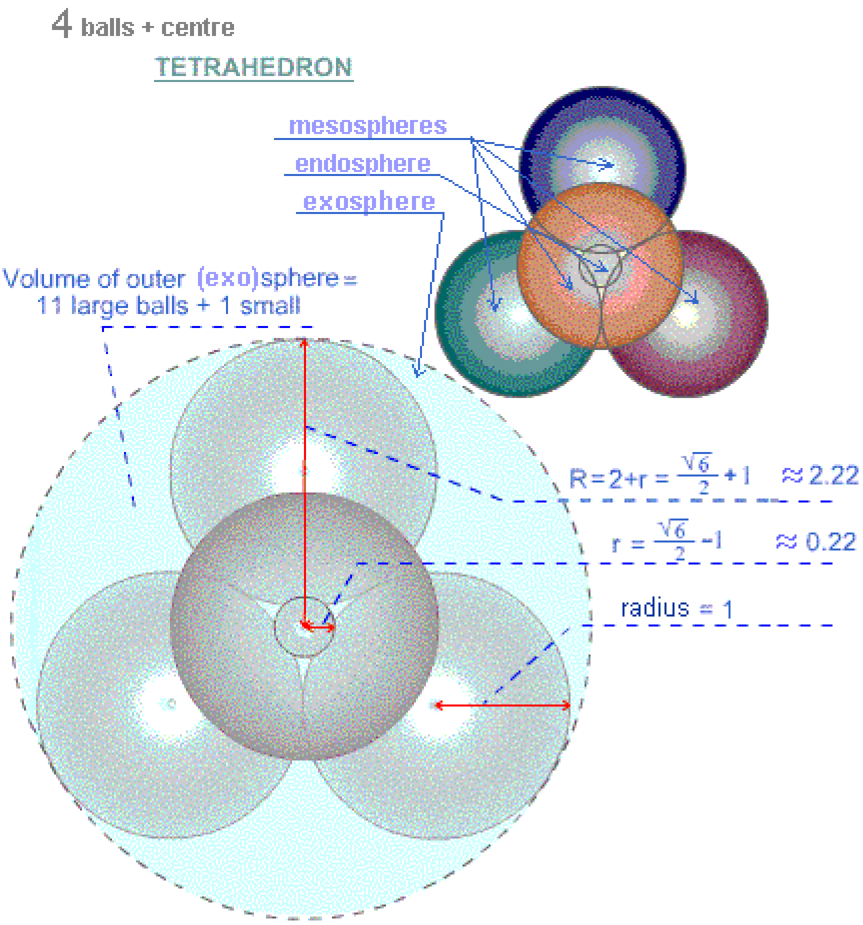
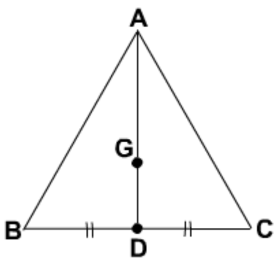


Theorems used:

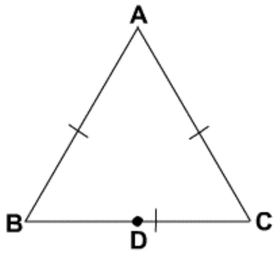
1) The "Centre" of any triangle (more properly the centroid), is one-third of the way up the medians of the triangle. (The median is the line joining a vertex to the mid-point of the opposite side).



The median AD has point G on it such that DG = one third AD

2) In 3-D, considering a regular tetrahedron, the centre of the tetrahedron will be vertically above the centre of the equilateral triangle which is its base. How far above this point? A quarter of the height of the tetrahedron.

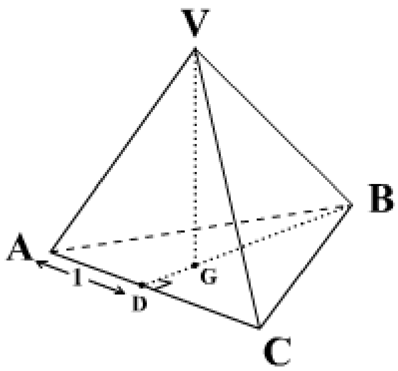
Proof of 1):



The "centre" of the equilateral triangle is the centre of mass (C.M.) point. So, imagine three particles, each of unit mass, placed at the vertices of the triangle. To find the C.M. of the triangle we may replace the particles of B and C by a particle of mass 2 units at D, which is mid-way between them. The C. M. of 2 units of D and 1 unit of A lies on AD and divides AD in the ratio 2:1, i.e. DG = one third DA.

Proof of 2): Is directly analogous to 1).

Proof proper:



V, A, B and C are the centres of 4 "mesospheres" that are touching each other, i.e. in the shape of a tetrahedron. Each has unit radius.

Set up a tetrahedron VABC, so that AB=BC=CA=AV=VC=VB=2.

BD is the median (of the equilateral triangle ABC). So BDC is a right-angled triangle.

Using Pythagoras' Theorem:

$\Rightarrow \quad BC^2 = BD^2 + DC^2$

$\Rightarrow \quad 2^2 = BD^2 + 1^2$

$\Rightarrow \quad BD^2 = 4 - 1 \qquad \Rightarrow \quad BD = \sqrt{3}$

Similarly DV = $\sqrt{3}$

Since DG = ⅓ DB (using result **1)** above)

Then BG = ⅔ $\sqrt{3}$

Consider the right-angled triangle VGB.

Again using Pythagoras: $VB^2 = VG^2 + GB^2 \quad \Rightarrow \quad 2^2 = VG^2 + \left(\frac{4}{9} \times 3\right)$

$\Rightarrow \quad 4 = VG^2 + \frac{4}{3}$

$\Rightarrow \quad VG^2 = 4 - \frac{4}{3} = \frac{8}{3}$

Hence, height of tetrahedron: $VG = \sqrt{\frac{8}{3}} = \frac{2\sqrt{2}}{\sqrt{3}} = \frac{2\sqrt{2}}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = \frac{2}{3}\sqrt{6}$

The centre of the tetrahedron,, is the centre of the “endosphere”, which is the sphere that would occupy the central space of the four mesospheres. Using result **2**) above,  is ¼ GV above the triangular base ABC, i.e. it is ¼ $\left(\frac{2}{3}\sqrt{6}\right)$ above the base = $\left(\frac{1}{6}\sqrt{6}\right)$ above the triangular base.

Now, start thinking in terms of the 4 “mesospheres”. The “bottom of the “top” mesosphere will be (VG – 1) = $\left(\frac{2}{3}\sqrt{6}-1\right)$ above the triangular base of the tetrahedron.

$\Rightarrow$ Radius of "endosphere", r, = $\left(\frac{2}{3}\sqrt{6}-1\right)-\frac{1}{6}\sqrt{6}$

$$\Rightarrow \quad r = \frac{1}{2}\sqrt{6}-1 \qquad (\approx 0.2247)$$

Define the “exosphere” as that sphere that encompasses the 4 mesospheres and touches all of them. Call its radius **R**.
Since the diameter of each mesosphere is 2, then clearly **R** = **r** + 2,
i.e. **R** = $\frac{1}{2}\sqrt{6}+1$ ($\approx$ 2.2247)

We can now calculate the volumes of the exosphere and endosphere, in particular the ratio **:** volume of exosphere **:** volume of endosphere.
From geometry, ratio of volumes = (ratio of radii)^3

$\Rightarrow$ volume of exosphere **:** volume of endosphere = $\left[\left(\frac{\sqrt{6}}{2}+1\right)/\left(\frac{\sqrt{6}}{2}-1\right)\right]^3 = \left(\frac{\sqrt{6}+2}{\sqrt{6}-2}\right)^3$

$$= \left(\frac{(\sqrt{6}+2)}{(\sqrt{6}-2)}\frac{(\sqrt{6}+2)}{(\sqrt{6}+2)}\right)^3 = \left(\frac{6+4\sqrt{6}+4}{6-4}\right)^3 = \left(\frac{10+4\sqrt{6}}{2}\right)^3 = \left(5+2\sqrt{6}\right)^3$$

$$= 5^3 + 3 \times 5^2(2\sqrt{6}) + 3\times5(2\sqrt{6})^2 + (2\sqrt{6})^3$$

$$= 125 + 150\sqrt{6} + 360 + 48\sqrt{6}$$

$$= \underline{485 + 198\sqrt{6}} \quad (\approx 970:1)$$

Also, volume of exosphere $= \frac{4}{3}\pi\left(\frac{\sqrt{6}}{2}+1\right)^3 = \frac{4}{3}\pi\left(\frac{\sqrt{6}+2}{2}\right)^3 = \frac{4}{3}\pi\left(\frac{6\sqrt{6}+3.6.2+3.\sqrt{6}.4+8}{8}\right)$

$$= \frac{\pi}{6}\left(18\sqrt{6}+44\right) = \underline{\frac{\pi}{3}\left(9\sqrt{6}+22\right)}$$

Similarly, volume of endosphere $= \underline{\frac{\pi}{3}\left(9\sqrt{6}-22\right)}$

From the above 2 results:

Volume of (exosphere – endosphere).

$$= \frac{\pi}{3}\left(9\sqrt{6}+22\right) - \frac{\pi}{3}\left(9\sqrt{6}-22\right) = \frac{\pi}{3}\cdot 44 = \frac{4\pi}{3}\cdot 11$$

= 11 x (volume of a mesosphere).

And further:

Volume of (exo – endo – mesospheres) = 7 x (volume of a mesosphere)

So we see that the space between the **exo**sphere and the **endo**sphere consists of 4 identifiable (basic) spheres, and the remaining space equivalent to 7 of these spheres.

For Sarossians:

These two divisions represent the two intervals – 4 and 7 on the Saros diagram. That is between 2 and 7 and the other 6 and 12.