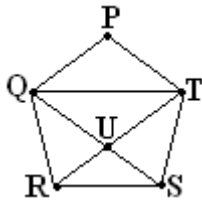


## PROOF OF ICOSAHEDRAL CENTRE

### Lemma 1

The 2 diagonals of a regular pentagon intersect each other in the ratio  $\emptyset : 1$ , where  $\emptyset$  is the golden mean defines as follows. Take a line AB and a point C on it. Then, if  $AB:AC = AC:CB$  we say that C divides AB in the golden ratio, and that each of the above ratios is as  $\emptyset : 1$ . The value of  $\emptyset$  is approximately 1.618.



By symmetry  $PQ//TR$  and  $PT//QS \Rightarrow PTUQ$  is a rhombus, in particular  $PT = QU$  — \*  
Consider the similar (Isosceles)  $\Delta$ s,  $\Delta RUS$  and  $\Delta QUT$

Then:  $\frac{QU}{US} = \frac{QT}{RS} = \frac{QS}{PT} = \frac{QS}{QU}$ , the last equivalence by \* above.

i.e.  $QU:US = QS : QU$  }  
similarly  $TU : UR = TR : TU$  } which establishes the lemma.

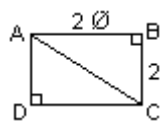
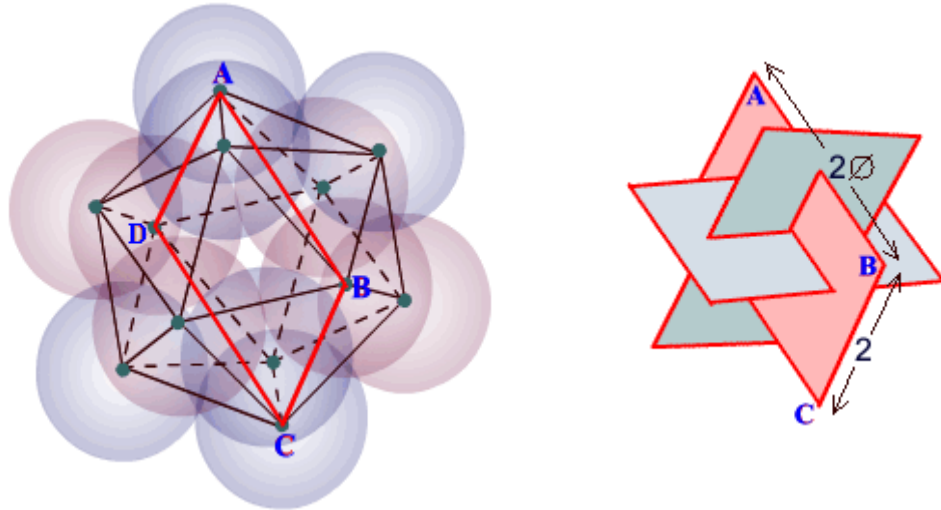
### Lemma 2

A diagonal of a regular pentagon of side 1 has length  $\emptyset$  .  
Let the above ratios be equal to  $\emptyset$ , i.e.  $QT:RS = \emptyset : 1$ .  
If  $RS = 1$ , then  $QT = \emptyset$  which establishes this lemma.

### Proof proper:

Let the spheres of the icosahedral grouping have unit radius. (Distinguish between the icosahedron and the icosahedral-grouping)

Within the icosahedron we can identify 3 planes which are mutually perpendicular, see diagram. Each plane is determined by 2 parallel edges of the icos. These 2 edges are 2 of the 4 edges of a rectangle. If the edge length of the icos. is 2 then this rectangle is 2 units wide and  $2\phi$  units long (using Lemma 2).



Using Pythagoras:  $AC^2 = 2^2 + (2\phi)^2 = 4 + 4\phi^2 = 4(1 + \phi^2)$

$$\Rightarrow AC = 2\sqrt{1 + \phi^2}$$

Now the diameter of this rectangle is the distance from an apex of the icos. to the opposite apex, i.e. it is the longest diameter of the icos. So, the radius of the circumscribing sphere of the icosahedron is  $\sqrt{1 + \phi^2}$ . In the icosahedral-grouping of spheres, this is the length of a line from the centre of the grouping to the centre of a ball (or the radius of a sphere through the centres of the 12 balls),

$\Rightarrow$  Radius **R** of circumscribing sphere of the icosahedral-grouping (exosphere) is  $\mathbf{R} = \sqrt{1 + \phi^2} + 1$ .

Similarly, the radius of the sphere that would fill the space inside the grouping (the endosphere, say) is

$$\mathbf{r} = \sqrt{1 + \phi^2} - 1$$

$$\Rightarrow \mathbf{Rr} = (\sqrt{1 + \phi^2} + 1)(\sqrt{1 + \phi^2} - 1) = \phi^2 \quad \text{--- (†)}$$

$$\Rightarrow \frac{\mathbf{R}}{\phi} = \frac{\phi}{\mathbf{r}}$$

But, (using the rectangle argument)  $2\emptyset$  is the diameter of a sphere passing through the points of contact of the 12 spheres, the “contact” sphere, i.e.  $\emptyset$  is its radius, say  $\mathbf{R}_1$ .

$$\Rightarrow \underline{\mathbf{R} : \mathbf{R}_1 = \mathbf{R}_1 : \mathbf{r}}$$

i.e. vol. of exo. : vol of contact = vol. of contact : vol. of endo.

$$\text{Using Lemma 1, } \emptyset^2 = \frac{QS}{QU} \cdot \frac{QU}{US} = \frac{QS}{US} = \frac{QU+US}{US} = \frac{QU}{US} + 1 = \emptyset + 1$$

$$\Rightarrow \mathbf{Rr} = \mathbf{1} + \emptyset, \text{ using } (\dagger) \text{ above.}$$

$$\Rightarrow \frac{\mathbf{R}}{\mathbf{1} + \emptyset} = \frac{1}{\mathbf{r}} \quad \text{i.e. } \underline{\mathbf{R} : (\mathbf{1} + \mathbf{R}_1) = \mathbf{1} : \mathbf{r}}$$

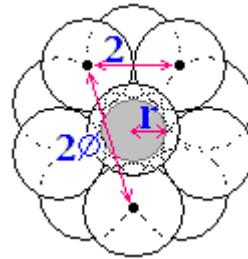
To calculate  $\emptyset$ , note that  $\emptyset^2 = \emptyset + 1$ , i.e.  $\emptyset^2 - \emptyset - 1 = 0$

$$\Rightarrow \left(\emptyset - \frac{1}{2}\right)^2 - \frac{5}{4} = 0 \quad \Rightarrow \left(\emptyset - \frac{1}{2}\right)^2 = \frac{5}{4} \quad \Rightarrow \emptyset - \frac{1}{2} = \sqrt{\frac{5}{4}} = \frac{\sqrt{5}}{2}$$

$$\Rightarrow \emptyset = \frac{\sqrt{5}}{2} + \frac{1}{2} \quad \Rightarrow \underline{\emptyset = \frac{\sqrt{5}+1}{2}}$$

$$\text{So, } \emptyset^2 + 1 = \emptyset + 2 = \frac{\sqrt{5}+1}{2} + 2 = \frac{\sqrt{5}+5}{2}$$

$$\Rightarrow \underline{\mathbf{r} = \frac{\sqrt{\sqrt{5}+5}}{2} - 1} \approx 0.902$$



Top view of the icos. grouping